An Introduction to Extractors

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Randomness

Why use randomized algorithms and protocols?

- Faster algorithms:
  e.g., PCP theorem (a slow computer can efficiently check computations done by a supercomputer)

- Better algorithms:
  Algorithms can have, in expectation, a better solution than the best deterministic algorithms

- Cryptography:
  Many protocols require random keys or challenges

Sources of Randomness

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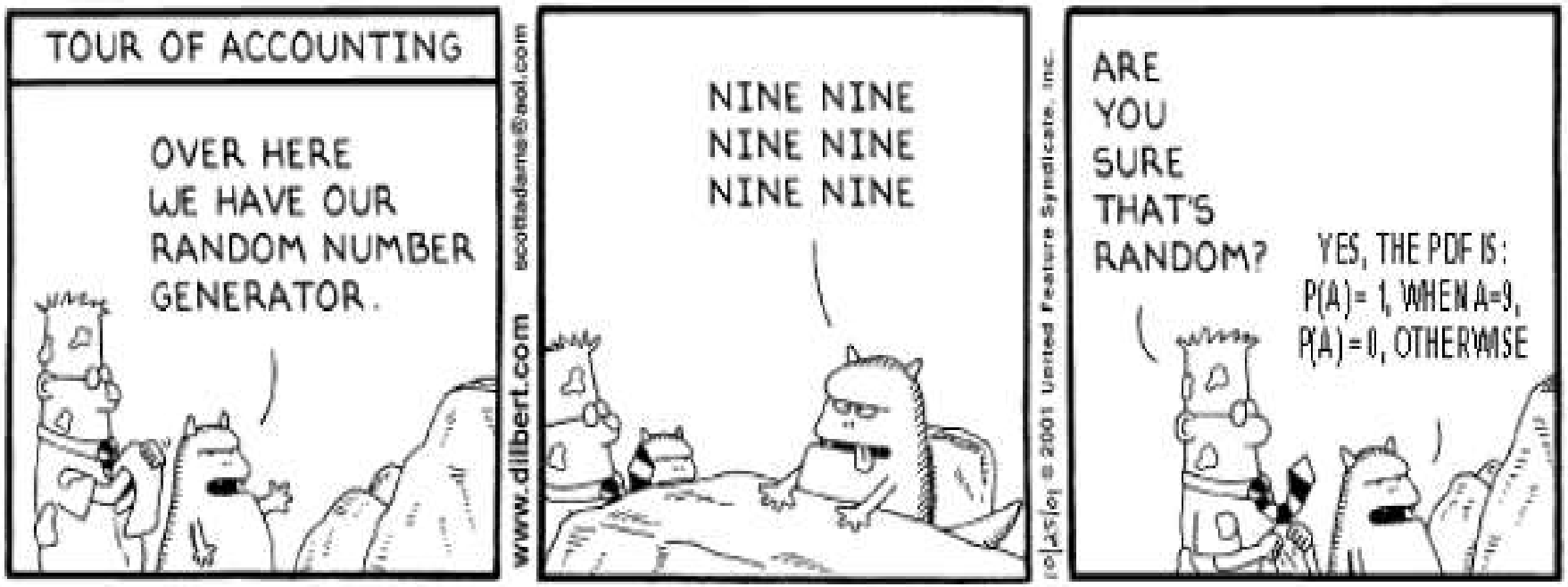

Sources of Randomness

- Coin tosses?
- Pseudo-Random Generator?
- Environment variables?
  Temperature, time, CPU state, ...
- Physical processes?
  Zener diode, Geiger counter, ...

Many sources are somewhat random, but not completely random.

An Extractor is a deterministic procedure that extracts truly random bits from such a weak source.

Measuring Randomness

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- A source is described by the probability distribution S of its output
- "Truly random bits": A sequence of uniformly distributed and independent bits
- Randomness of a source S is measured by Min-Entropy:

$$H_\infty(S) = \min_x \left\{ \log \left(\frac{1}{\Pr_S[x]} \right) \right\}$$

- A k-source is a source S with $H_\infty(S) \geq k$

Comparing Distributions

Extractor's output distribution O should be "close to uniform"

Output of extractor is compared to uniform distribution with statistical difference Δ:

$$\Delta(P,Q) := \frac{1}{2}\sum_x \left| \Pr_P(x) - \Pr_Q(x) \right|$$

O is ϵ-close to uniform distribution U iff

$$\Delta(O,U) \leq \epsilon$$

Extractor: Simple Example

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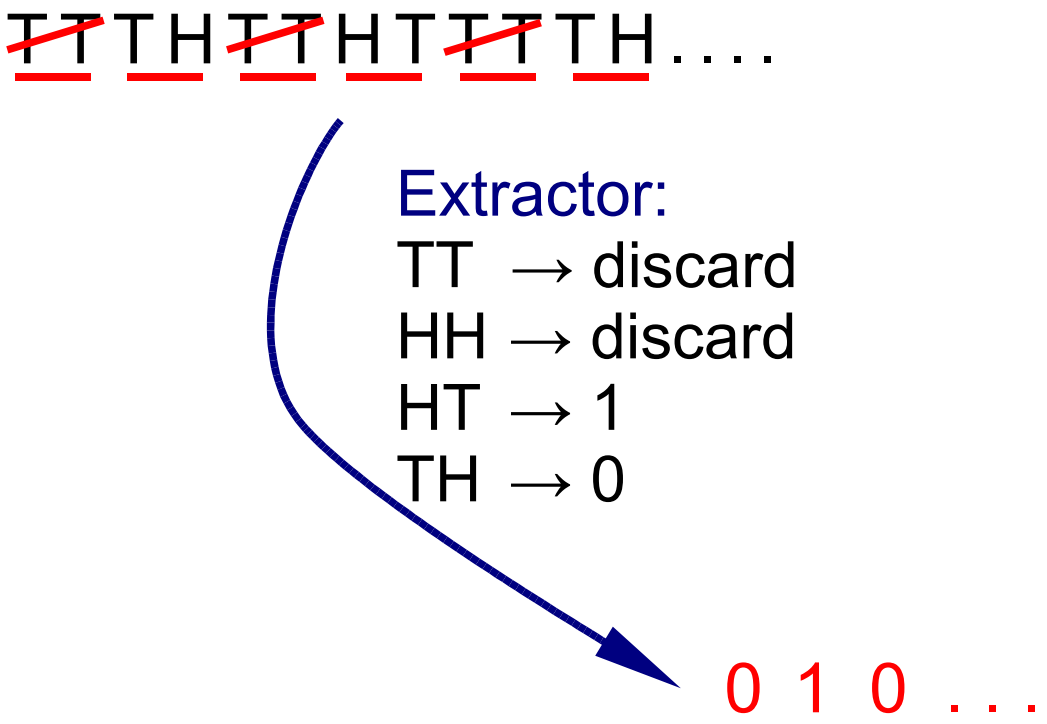
We have: Sequence of {H,T} by biased coin:
$\Pr[H] = h$, $\Pr[T] = t$, $t \neq h$

We want: Uniformly distributed {0,1}-Sequence.



Output is Uniform, since Pr[HT] = Pr[TH] = ht

Extractors: Formal Definition

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$$Ext : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$$

so that for an arbitrary source S on $\{0,1\}^n$ with $H_\infty(S) \geq k$, $Ext(S, U_d)$ is ϵ-close to U_m.

Weak randomness
(n Bits)

0001010001...

Extractor

Random output
(m Bits)

0010...

Truly random seed
(d Bits)

Why do we need a Seed?

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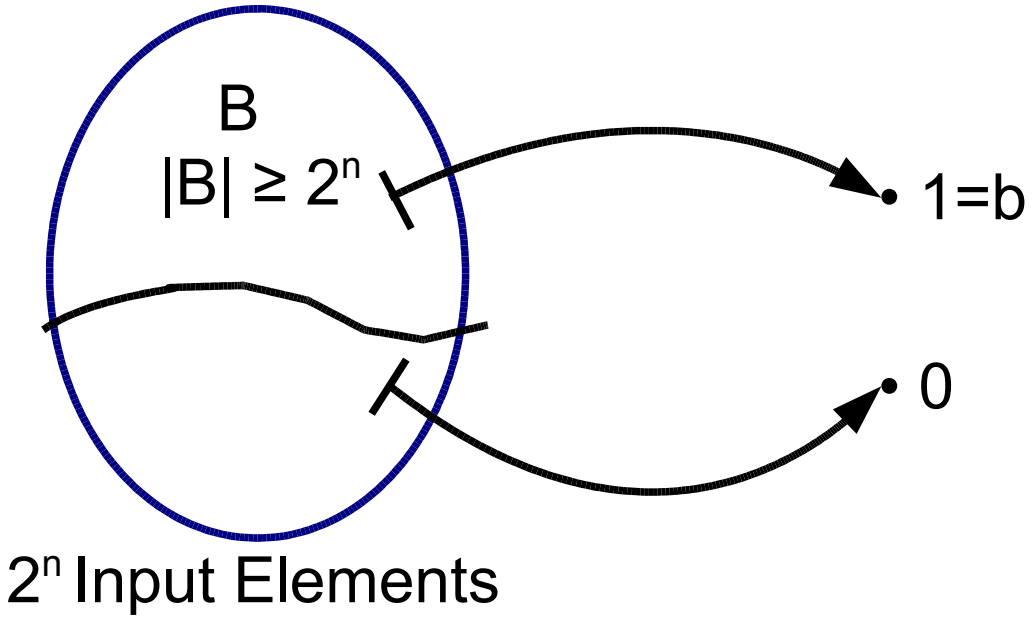
Claim: for any $Ext : \{0,1\}^n \rightarrow \{0,1\}$, $\exists$ (n-1)-source S s.t. $Ext(S) = const$.

B, b as defined in figure. Let S be the uniform distribution on B

$\Rightarrow H_\infty(S) \geq \log \frac{1}{1/2^{n-1}} = n - 1$

$\Rightarrow Ext(S) = b = const$



Optimal Extractors

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The following lower bounds apply to extractors for arbitrary sources. (i.e., any n, $k \leq n$, $\epsilon > 0$.)

- Seed length:
  $d \geq \log(n-k) + 2\log(\frac{1}{\epsilon}) - O(1)$
  [Nisan, Zuckermann, 96]

- Entropy loss:
  no non-trivial extractor can extract all randomness present in a source.
  $k + d - m \geq 2\log(\frac{1}{\epsilon}) - O(1)$
  [Radhakrishnan, Ta-Shma, 00]

Asymptotically optimal constructions are known, although not for arbitrary k.

Simple Extractor Construction

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Construction is based on a universal family of hash functions:

$$H = \left\{h : \{0,1\}^n \to \{0,1\}^l\right\} \text{ s.t. } \Pr_{h \in H, x \neq y}[h(x) = h(y)] \leq \frac{1}{2^l}$$

Example: $H = \{H_{ab}\} = last\ l\ bits\ of\ ax + b$ in GF(2^n).

$$Ext(x,h) := h||h(x)$$

is Extractor with Error ϵ and min-entropy-threshold k if $l = k - 2\log(\frac{1}{\epsilon}) - O(1)$.

Proof: show that $H||H(X)$ is ϵ-close to $U_d \times U_l$.

Proof(1): Collision Prob. is small

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Claim: $cp\,(H||H(X)) \simeq cp\,(U_d \times U_l)$.

Facts:

$|H| = 2^d$, because d bits of seed define 2^d hash functions.

$H_\infty(X) \geq k \Rightarrow \Pr[X = X'] \leq \frac{1}{2^k}$.

$\Pr\,[H(X) = H(X')|X \neq X'] = \frac{1}{2^l}$ by definition of universal hash functions.

Proof: $cp\,(H||H(X)) = \Pr\,[(H||H\,(X)) = (H'||H'\,(X'))]$

$$= \frac{1}{|H|}\left(\Pr\,[X = X'] + \Pr\,[H(X) = H(X')|X \neq X']\,\Pr\,[X \neq X']\right)$$

$$= \frac{1}{2^d}\left(\frac{1}{2^k} + \frac{1}{2^l}\left(1 - \frac{2}{2^k}\right)\right) \leq \frac{1}{2^d}\left(\frac{1}{2^k} + \frac{1}{2^l}\right) = \frac{1}{2^d 2^l}\left(1 + \epsilon^2\right)$$

(with $\epsilon = \frac{2^l}{2^k}$)

Proof(2): Error is small

Let $U := U_d \times U_l$, $V := H||H(X)$. Let Θ be the output domain.

Claim: $cp(V) \simeq cp(U) \Rightarrow V$ is ϵ-close to U

Proof: $\Delta(V,U) = \frac{1}{2}\sum_{x\in\Theta} |\Pr_U[x] - \Pr_V[x]|$

$$\leq \frac{\sqrt{|\Theta|}}{2}\sqrt{\sum_{x\in\Theta} |\Pr_V[x] - \Pr_U[x]|^2}$$

$$= \frac{\sqrt{|\Theta|}}{2}\sqrt{\sum_{x\in\Theta} \left(\Pr_V[x]^2 - 2\frac{\Pr_V[x]}{|\Theta|} + \frac{1}{|\Theta|^2}\right)}$$

$$= \frac{\sqrt{|\Theta|}}{2}\sqrt{cp(V) - \frac{1}{|\Theta|}}$$

$$\leq \frac{\sqrt{2^d 2^l}}{2}\sqrt{\frac{\epsilon^2}{2^d 2^l}} = \frac{\epsilon}{2}$$

Privacy Amplification

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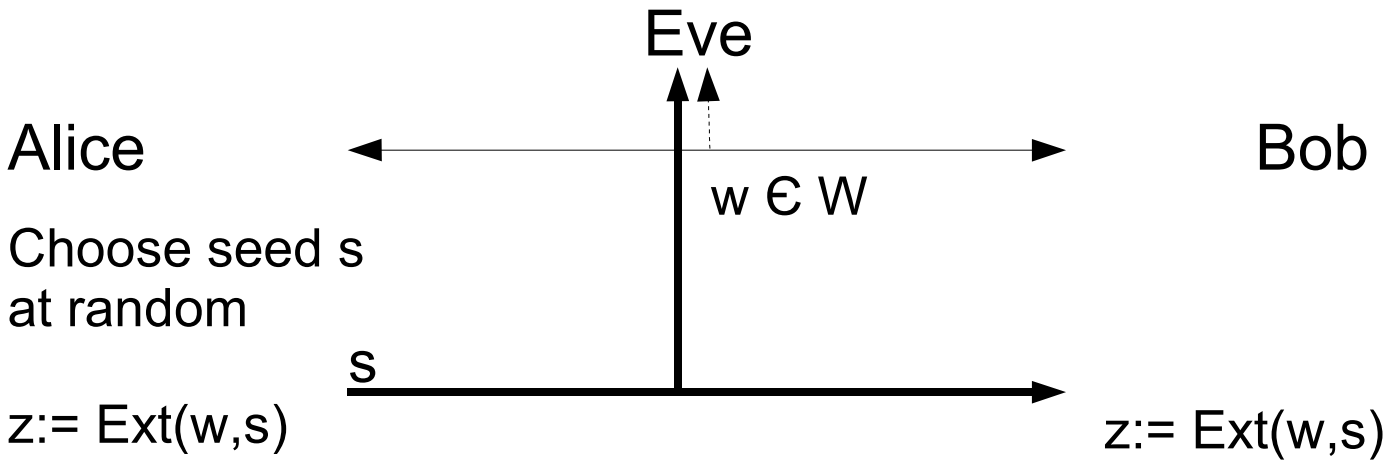
Setting: Alice and Bob know variable w, about which Eve has partial knowledge, i.e. Eve knows v according to a distribution P_{VW}. v contains at most t bits of information about w. Assume $W = U_{2^n}$.

Alice & Bob want a function f s.t. Eve has almost no information about $z = f(w)$.

Solution: Extractor function!



Privacy Amplification: Proof

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Claim: $\Delta((Z||V),(U \times V)) \leq \epsilon$

Proof: $\Delta((Z||V),(U \times V)) = \ldots$
$= \sum_v \Pr_V(v) * \Delta((Z|V=v),U)$

Because Eve knows at most t bits about W, $H_\infty(W|V=v) \geq n-t$.

$\Rightarrow$we can use an Extractor with min-entropy-threshold $n-t$ and error ϵ to extract a random Z from W
$\Rightarrow \forall v: \Delta((Z|V=v),U) \leq \epsilon$
$\Rightarrow \sum \Pr_V(v) * \Delta((Z|V=v),U) \leq \epsilon$

Whatever algorithm Eve uses to discover z, her advantage over blindly guessing is not more than ϵ!

Thanks for your attention!

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Further reading:

- Salil Vadhan, Lectures on Pseudorandomness (12-14).
  `http://www.courses.fas.harvard.edu/~cs225/` .
- Ronen Shaltiel. Recent developments in extractors.
  `http://www.cs.haifa.ac.il/~ronen/online_papers/survey.ps` .
- Valentine Kabanets, Lectures on Pseudorandomness (12-15).
  `www.cs.sfu.ca/~kabanets/cmpt881/lec/` .
- Slides: `http://n.ethz.ch/student/fuchsd/extractors.pdf` .